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New relations between Hamiltonian and Lagrangian constraints

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Abstract. Different procedures for obtaining all the Lagrangian constraints from the Hamiltonian constraints are studied. The connection of arbitrary functions in both Hamiltonian and Lagrangian formalisms is also studied. As a consequence we give a way to pass directly from the final Hamilton-Dirac equations to the Euler-Lagrange ones.

1. Introduction

In previous papers [1, 2], we obtained new results concerning the equivalence between the Lagrangian and Hamiltonian formalisms for constrained systems. The equivalence between both formalisms was shown in the sense that, given a solution $q(t)$ of Euler-Lagrange equations of motion, the functions $q(t)$ and $p(t) = (\partial L/\partial \dot{q})(q(t), \mathrm{d}q/\mathrm{d}t)$ are solutions of the Hamiltonian-Dirac equations of motion and vice versa. A constructive algorithm—different, but equivalent to, the standard ones (see, for example [3])—was given to build up, step by step, the set of Hamiltonian constraints and a new algorithm for the Lagrangian case, which has its roots in the Hamiltonian one, was introduced.

The present paper gives some new results on the lines of [1, 2]. In the following we will use the same conventions and notation. In § 2 some preliminary aspects are treated. In § 3 some new results on the connection between Hamiltonian and Lagrangian constraints are given. In § 4 the relations between the arbitrary functions of both formalisms are established. In § 5 we give a new procedure to pass from the final Hamilton-Dirac equations of motion, on the surface of constraints, to the final Euler-Lagrange ones. Section 6 is devoted to some conclusions.

2. Preliminary aspects

When one starts from a singular Lagrangian, evolution is subjected to a certain ambiguity due to the appearance of arbitrary functions either in the Lagrangian or Hamiltonian formalism. Let us first consider the Hamiltonian case. The generator of time evolution on the surface M_0 of primary constraints in cotangent space T^*Q is

$$D_{\mathrm{H}} = \{-, \mathscr{H}_{\mathrm{c}}\} + u^{\nu}(t)\{-, \phi_{\nu}^{(0)}\} \tag{1}$$

where $\mathscr{H}_{\mathrm{c}}$ is one of the possible canonical Hamiltonians [3], defined in T^*Q and all giving the same function in M_0. $\phi_{\nu}^{(0)}$, $\nu = 1, \ldots, m_1$, are the primary constraints and u^{ν} are, in principle, arbitrary functions of time. It is worthwhile remarking that these functions have a definite form, $v^{\mu}(q, \dot{q})$, on the tangent space TQ (see [2], equation (2.17))† and some of them will be determined as functions of T^*Q by requiring $D_{\rm H}$ to be tangent to the surface of constraints. Generally, some of these functions will remain undetermined at the end of the consistency algorithm.

On the other hand, the evolution operator in the Lagrangian case, defined on the surface S_1 of TQ by the primary Lagrangian constraints, is [2]

$$D_{\rm L} = \mathring{D} + \eta^{\nu}(t)\Gamma_{\nu} \tag{2}$$

where

$$\mathring{D} = \dot{q}\frac{\partial}{\partial q} + (M\alpha)\frac{\partial}{\partial \dot{q}} \tag{3}$$

$$M = FL^* \frac{\partial^2 \mathscr{H}_{\rm c}}{\partial p\, \partial p} + v^{\mu}(q, \dot{q}) FL^* \frac{\partial^2 \phi^{(0)}_{\mu}}{\partial p\, \partial p} \tag{4}$$

$$\alpha = \frac{\partial L}{\partial q} - \dot{q}\frac{\partial^2 L}{\partial q\, \partial \dot{q}} \qquad \Gamma_{\nu} = \gamma_{\nu}\frac{\partial}{\partial \dot{q}} \qquad \gamma_{\nu} = FL^* \frac{\partial \phi^{(0)}_{\nu}}{\partial p} \tag{5}$$

FL^* being the pull-back of the fibre derivative of the Lagrangian.

Let us observe that, owing to the second-order character of the operator $D_{\rm L}$ (the coefficient of $\partial/\partial q$ is just $\dot{q}$), every integral curve $(q(t), \dot{q}(t))$ of $D_{\rm L}$ will satisfy $\mathrm{d}q/\mathrm{d}t = \dot{q}(t)$.

$\eta^{\nu}(t)$ are, in principle, arbitrary functions of time. A certain number of these functions will be determined requiring $D_{\rm L}$ to be tangent to the surface of constraints. It is shown in [2] that this number is just the same as the one in the Hamiltonian case.

We have seen, therefore, that the evolution operators $D_{\rm H}$, $D_{\rm L}$ contain a certain degree of arbitrariness. It will be useful to consider a third evolution operator which has no ambiguity at all. This operator, K [1, 2, 4], takes a function in cotangent space and gives its time derivative as a function in tangent space:

$$\begin{aligned} &K : \Lambda^0(T^*Q) \to \Lambda^0(TQ) \\ &K = \dot{q} FL^* \circ \frac{\partial}{\partial q} + \frac{\partial L}{\partial q} FL^* \circ \frac{\partial}{\partial p}. \end{aligned} \tag{6}$$

So we have the scheme (not commutative!)

$$\begin{array}{ccc} \Lambda^0(TQ) & \xleftarrow{\;FL^*\;} & \Lambda^0(T^*Q) \\ {\scriptstyle D_{\rm L}}\downarrow & \swarrow{\scriptstyle K} & \downarrow{\scriptstyle D_{\rm H}} \\ \Lambda^0(TQ) & \xleftarrow{\;FL^*\;} & \Lambda^0(T^*Q). \end{array} \tag{7}$$

We shall now derive some relations connecting the different operators $D_{\rm L}$, $D_{\rm H}$, K, in (7).

From the results obtained in [2] (see, in particular, the equivalence of (2.2) and (2.17) of [2] and also equation (2.22) of [2]) we know the following identities in

† This statement has the following meaning. Let the determined functions in TQ be $v^{\mu}(q, \dot{q})$. Then, if we have an integral curve $(q(t), p(t))$ of $D_{\rm H}$ on M_0 that corresponds to some given functions $u^{\mu}(t)$, the following relation holds: $v^{\mu}(q(t), \mathrm{d}q/\mathrm{d}t) = u^{\mu}(t)$, $\mu = 1, \ldots, m_1$.

tangent space:

$$\dot{q} = FL^* \frac{\partial \mathcal{H}_c}{\partial p} + v^\mu(q, \dot{q}) FL^* \frac{\partial \phi_\mu^{(0)}}{\partial p} \tag{8}$$

$$-\frac{\partial L}{\partial q} = FL^* \frac{\partial \mathcal{H}_c}{\partial q} + v^\mu(q, \dot{q}) FL^* \frac{\partial \phi_\mu^{(0)}}{\partial q}. \tag{9}$$

Now, given $g \in \Lambda^0(T^*Q)$ we can calculate

$$FL^*\{g, \mathcal{H}_c\} = FL^* \frac{\partial g}{\partial q} FL^* \frac{\partial \mathcal{H}_c}{\partial p} - FL^* \frac{\partial g}{\partial p} FL^* \frac{\partial \mathcal{H}_c}{\partial q}$$

and using (8) and (9)

$$\begin{aligned} FL^*\{g, \mathcal{H}_c\} &= FL^* \frac{\partial g}{\partial q}\left(\dot{q} - v^\mu(q, \dot{q}) FL^* \frac{\partial \phi_\mu^{(0)}}{\partial p}\right) \\ &\quad - FL^* \frac{\partial g}{\partial p}\left(-\frac{\partial L}{\partial q} - v^\mu(q, \dot{q}) FL^* \frac{\partial \phi_\mu^{(0)}}{\partial q}\right) \\ &= \dot{q} FL^* \frac{\partial g}{\partial q} + \frac{\partial L}{\partial q} FL^* \frac{\partial g}{\partial p} - v^\mu(q, \dot{q}) FL^*\{g, \phi_\mu^{(0)}\} \end{aligned}$$

so

$$Kg = FL^*\{g, \mathcal{H}_c\} + v^\mu(q, \dot{q}) FL^*\{g, \phi_\mu^{(0)}\} \tag{10}$$

or

$$K = FL^* \circ \{-, \mathcal{H}_c\} + v^\mu FL^* \circ \{-, \phi_\mu^{(0)}\}.$$

We recognise on the RHS of (10) the pullback of $D_{\mathrm{H}} g$ *but* with the substitution of the arbitrary functions $u^\mu(t)$ by its values on TQ: $v^\mu(q, \dot{q})$.

Considering again $g \in \Lambda^0(T^*Q)$, we also get

$$\frac{\partial^{\mathrm{L}}}{\partial q} FL^* g = FL^* \frac{\partial^{\mathrm{H}} g}{\partial q} + \frac{\partial \mathfrak{P}}{\partial q} F^* \frac{\partial q}{\partial p}$$

where we distinguish between $\partial^{\mathrm{L}}/\partial q$ acting on $\Lambda^0(TQ)$ and $\partial^{\mathrm{H}}/\partial q$ acting on $\Lambda^0(T^*Q)$. $\mathfrak{P}$ stands for $\partial L/\partial \dot{q}$.

Thus we have

$$\frac{\partial^{\mathrm{L}}}{\partial q} \circ FL^* = FL^* \circ \frac{\partial^{\mathrm{H}}}{\partial q} + \frac{\partial \mathfrak{P}}{\partial q} FL^* \circ \frac{\partial}{\partial p} \tag{11}$$

and in an analogous way

$$\frac{\partial}{\partial \dot{q}} \circ FL^* = W FL^* \circ \frac{\partial}{\partial p} \qquad W = \frac{\partial^2 L}{\partial \dot{q}\, \partial \dot{q}}. \tag{12}$$

With these relations in mind, and taking into account (4) and (A1),

$$M^{ik} \frac{\partial}{\partial \dot{q}^k} \circ FL^* = M^{ik} W_{kj} FL^* \circ \frac{\partial}{\partial p_j} = \left(\delta^i_j - \frac{\partial v^\nu}{\partial \dot{q}^j} \gamma^i_\nu\right) FL^* \circ \frac{\partial}{\partial p^j}$$

and therefore

$$\alpha_i M^{ik} \frac{\partial}{\partial \dot{q}^k} \circ FL^* = \left(\alpha_j - (\alpha \gamma_\nu) \frac{\partial v^\nu}{\partial \dot{q}^j}\right) FL^* \circ \frac{\partial}{\partial p_j} \tag{13}$$

and

$$\dot{q}^i \frac{\partial^{\mathrm{L}}}{\partial q^i} \circ FL^* = \dot{q}^i FL^* \circ \frac{\partial^{\mathrm{H}}}{\partial q} + \dot{q}^i \frac{\partial \mathfrak{P}_j}{\partial q^i} FL^* \circ \frac{\partial}{\partial p_j}. \tag{14}$$

We can now calculate

$$D_{\mathrm{L}} \circ FL^* = \dot{q}^i \frac{\partial^{\mathrm{L}}}{\partial q^i} \circ FL^* + \alpha_i M^{ik} \frac{\partial}{\partial \dot{q}^k} \circ FL^* + \eta^\nu(t) \Gamma_\nu \circ FL^*$$

but $\Gamma_\nu \circ FL^* = 0$ because Γ_ν, $\nu = 1, \ldots, m_1$, span ker FL_*. Then, using (13) and (14), we get

$$\begin{aligned} D_{\mathrm{L}} \circ FL^* &= \dot{q}^i FL^* \circ \frac{\partial^{\mathrm{H}}}{\partial q^i} + \left(\alpha_j - (\alpha\gamma_\nu) \frac{\partial v^\nu}{\partial \dot{q}^j} + \dot{q}^i \frac{\partial \mathfrak{P}_j}{\partial q^i} \right) FL^* \circ \frac{\partial}{\partial p_j} \\ &= \dot{q} FL^* \circ \frac{\partial^{\mathrm{H}}}{\partial q} + \frac{\partial L}{\partial q} FL^* \circ \frac{\partial}{\partial p} - (\alpha\gamma_\nu) \frac{\partial v^\nu}{\partial \dot{q}} FL^* \circ \frac{\partial}{\partial p_j} \end{aligned}$$

or

$$K = D_{\mathrm{L}} \circ FL^* + (\alpha\gamma_\nu) \frac{\partial v^\nu}{\partial \dot{q}} FL^* \circ \frac{\partial}{\partial p}. \tag{15}$$

If we recall (see [2], equation (4.4)) that $(\alpha\gamma_\nu)$ are the primary Lagrangian constraints which define the surface S_1, we have

$$Kg \underset{S_1}{=} D_{\mathrm{L}}(FL^* g) \qquad g \in \Lambda^0(T^*Q). \tag{16}$$

Equations (10) and (15) show how the diagram (7) works.

3. New expressions for the Lagrangian constraints

The equivalence of the Euler–Lagrange and Hamilton–Dirac equations yields immediately the result that $FL^*\phi = 0$ is a Lagrangian constraint $\forall \phi \in \mathscr{C}$, $\mathscr{C}$ being the set of Hamiltonian constraints.

All these Lagrangian constraints are projectable (they come from a Hamiltonian constraint) and therefore they conserve the foliation of TQ induced by FL. Let us comment on this fact briefly. Given a point $(q_0, p_0) \in M_0 \subset T^*Q$, the anti-image $FL^{-1}(q_0, p_0)$ is a sheet in TQ. This sheet is defined by q_0 and by the m_1-parametric family

$$\dot{q}_0(\beta^\mu) = \{q, \mathscr{H}_c\}(q_0, p_0) + \beta^\mu \{q, \phi_\mu^{(0)}\}(q_0, p_0). \tag{17}$$

This sheet can also be seen as a maximal integral surface of ker FL^*, spanned by the fields Γ_ν, $\nu = 1, \ldots, m_1$. Since $\Gamma_\nu \circ FL^* = 0$ we see that the sheets are either entirely inside or entirely outside the surface defined by constraints $FL^*\phi$, $\phi \in \mathscr{C}$.

But, on the other hand, the evolution operator in the final surface $M_{\mathrm{F}} \subset T^*Q$ of Hamiltonian constraints is (see [2], equation (3.42))

$$D_{\mathrm{H_F}} \underset{M_{\mathrm{F}}}{=} \{-, \mathscr{H}_{\mathrm{F}}\} + u^{\nu_{\mathrm{f}}}(t) \{-, \phi_{\nu_{\mathrm{f}}}^{(0)}\} \tag{18}$$

where $\mathscr{H}_{\mathrm{F}}$ is the first class Hamiltonian and $\phi_{\nu_{\mathrm{f}}}^{(0)}$, $\nu_{\mathrm{f}} = 1, \ldots, r$, are the final primary first class constraints. Therefore, given $(q_0, p_0) \in M_{\mathrm{F}}$, the possible associated initial conditions in TQ are q_0 and the r parametric family:

$$\dot{q}_0(\beta^{\nu_{\mathrm{f}}}) = \{q, \mathscr{H}_{\mathrm{F}}\}(q_0, p_0) + \beta^{\nu_{\mathrm{f}}} \{q, \phi_{\nu_{\mathrm{f}}}^{(0)}\}(q_0, p_0). \tag{19}$$

Comparison of (17) and (19) (we can substitute $\mathscr{H}_c$ in (17) by $\mathscr{H}_F$ which simply corresponds to a change of parameters β^{μ}) shows that we must reduce the sheet (17) to (19) in order to determine the surface of Lagrangian constraints. This fact shows that $FL^*\phi=0$, $\phi\in\mathscr{C}$, is not the whole set of Lagrangian constraints. We shall now derive the constraints which are missing.

The first class Hamiltonian $\mathscr{H}_F$ appearing in (19) is [2]

$$\mathscr{H}_F=\mathscr{H}_c+\lambda^{\bar{\mu}_f}(q,p)\phi^{(0)}_{\bar{\mu}_f} \tag{20}$$

where $\phi^{(0)}_{\bar{\mu}_f}$, $\bar{\mu}_f=r+1,\ldots,m_1$, are the final primary second class constraints and $\lambda^{\bar{\mu}_f}(q,p)$ are a canonical determination of the (in principle, arbitrary) functions $u^{\bar{\mu}_f}(t)$ of (1).

So, keeping (q_0,p_0) in M_F, the possible velocities that give, together with q_0, a well posed initial condition in TQ are those of (19). Using the determination in TQ of parameters β^{μ_f}, $v^{\mu_f}(q,\dot{q})$, we get the implicit equation for $\dot{q}_0(\beta^{\mu_f})$:

$$\dot{q}_0=\{q,\mathscr{H}_c\}(q_0,p_0)+\lambda^{\bar{\mu}_f}(q_0,p_0)\{q,\phi^{(0)}_{\bar{\mu}_f}\}(q_0,p_0)+v^{\mu_f}(q_0,\dot{q}_0)\{q,\phi^{(0)}_{\mu_f}\}(q_0,p_0)$$

or

$$\dot{q}_0=FL^*\{q,\mathscr{H}_c\}(q_0,\dot{q}_0)+(FL^*\lambda^{\bar{\mu}_f})(q_0,\dot{q}_0)\gamma_{\bar{\mu}_f}(q_0,\dot{q}_0)+v^{\mu_f}(q_0,\dot{q}_0)\gamma_{\mu_f}(q_0,\dot{q}_0). \tag{21}$$

Comparing (21) with the identity (8) at the point $(q_0,\dot{q}_0)$ we obtain

$$0=(FL^*\lambda^{\bar{\mu}_f}(q_0,\dot{q}_0)-v^{\bar{\mu}_f}(q_0,\dot{q}_0))\gamma_{\bar{\mu}_f}(q_0,\dot{q}_0).$$

But the vectors $\gamma_{\bar{\mu}_f}(q_0,\dot{q}_0)$, $\bar{\mu}_f=r+1,\ldots,m_1$, are all independent. Therefore

$$\chi^{\bar{\mu}_f}(q_0,\dot{q}_0):=FL^*\lambda^{\bar{\mu}_f}(q_0,\dot{q}_0)-v^{\bar{\mu}_f}(q_0,\dot{q}_0)=0.$$

Thus the functions $\chi^{\bar{\mu}_f}$ are Lagrangian constraints.

These functions produce new restrictions on TQ that were not taken into account by $FL^*\phi=0$, $\forall\phi\in\mathscr{C}$. This is because $\chi^{\bar{\mu}_f}$ cut the sheets $FL^{-1}(q,p)$ whereas $FL^*\phi=0$ does not. The point is the following: recalling $\Gamma_\mu\circ FL^*=0$, we have $\Gamma_\mu\chi^{\bar{\mu}_f}=-\delta^{\bar{\mu}_f}_{\mu}$ ($\Gamma_\mu v^\nu=\delta^\nu_\mu$ comes from applying Γ_μ to both sides of (8)). Since the sheets $FL^{-1}(q,p)$ are the integral surfaces of Γ_μ, $\mu=1,\ldots,m_1$, we see that surfaces $\chi^{\bar{\mu}_f}=0$ cut these sheets and, moreover, that they are all independent.

We now need to prove that $FL^*\phi=0$, $\forall\phi\in\mathscr{C}$, and $\chi^{\bar{\mu}_f}=0$, $\bar{\mu}_f=r+1,\ldots,m_1$, are *all* the Lagrangian constraints. We bear in mind that, if a point $(q_0,\dot{q}_0)$ belongs to the surface of Lagrangian constraints, it can be taken as the initial condition for the trajectories which are solutions of Euler-Lagrange equations. The converse is obvious: an initial condition always belongs to the surface of constraints.

Now consider a point $(q_0,p_0)\in M$. By the same reasons given above, it can be used as the initial condition to solve Hamilton-Dirac equations. The first half of the HD equations is (19)

$$\dot{q}_0(\beta^{\mu_f})=\{q,\mathscr{H}_F\}(q_0,p_0)+\beta^{\mu_f}\{q,\phi^{(0)}_{\mu_f}\}(q_0,p_0).$$

Since we know from [1, 2] that there is a complete equivalence between solutions in either the Hamiltonian or Lagrangian formalism, the points $(q_0,\dot{q}_0(\beta^{\mu_f}))$ are the possible Lagrangian initial conditions whereas (q_0,p_0) remains in M. These points define, therefore, the surface of Lagrangian constraints.

What restrictions do these point satisfy? First, they have to be in $FL^{-1}(M_F)$ because points satisfying (19) are a subset of those satisfying (17). This condition is just $FL^*\phi=0$, $\forall\phi\in\mathscr{C}$. But the only velocities $\dot{q}_0(\beta^{\mu_f},\beta^{\bar{\mu}_f})$ admissible as initial Lagrangian conditions are those of (19), i.e. when $\beta^{\bar{\mu}_f}=0$. This requirement is equivalent to imposing the constraints $\chi^{\bar{\mu}_f}=0$.

In fact

$$\chi^{\bar{\mu}_f}(q_0, \dot{q}_0(\beta^{\mu_f}, \beta^{\bar{\mu}_f}))$$
$$= \chi^{\bar{\mu}_f}(q_0, \dot{q}_0(0)) + \beta^{\mu_f}(\Gamma_{\mu_f}\chi^{\bar{\mu}_f})(q_0, \dot{q}_0(0)) + \beta^{\bar{\nu}_f}(\Gamma_{\bar{\nu}_f}\chi^{\bar{\mu}_f})(q_0, \dot{q}_0(0))$$

with no higher-order terms in β because, from $\Gamma_\mu \chi^{\bar{\mu}_f} = -\delta_\mu^{\bar{\mu}_f}$, we derive

$$\Gamma_\mu \Gamma_\nu \chi^{\bar{\mu}_f} = \Gamma_\mu(-\delta_\nu^{\bar{\mu}_f}) = 0.$$

Since $\chi^{\bar{\mu}_f}(q_0, \dot{q}(0)) = 0$ because $\chi^{\bar{\mu}_f}$ are Lagrangian constraints and $\dot{q}_0(0)$ is described by (19), we have

$$\chi^{\bar{\mu}_f}(q_0, \dot{q}_0(\beta^{\mu_f}, \beta^{\bar{\mu}_f})) = -\beta^{\mu_f}\delta_{\mu_f}^{\bar{\mu}_f} - \beta^{\bar{\nu}_f}\delta_{\bar{\nu}_f}^{\bar{\mu}_f} = -\beta^{\bar{\mu}_f}.$$

We have seen, therefore, that the initial Lagrangian conditions only have the restrictions $FL^*\phi = 0$, $\forall \phi \in \mathscr{C}$ and $\chi^{\bar{\mu}_f}=0$. Thus we have proved the following.

Theorem 1. All the Lagrangian constraints can be written in the form:

$$FL^*\phi = 0 \qquad \forall \phi \in \mathscr{C} \tag{22}$$

$$\chi^{\bar{\mu}_f} := FL^*\lambda^{\bar{\mu}_f} - v^{\bar{\mu}_f} = 0 \qquad \bar{\mu}_f = r+1, \ldots, m_1. \tag{23}$$

Let us observe, by the way, that constraints (22)—which are the projectable constraints—are the ones which appear (see Gotay and Nester [3]) when the second-order condition is eliminated from the EL equations. Therefore the second-order condition is responsible for the non-projectable constraints (23).

It is now easy to calculate the number of Lagrangian constraints. Let us consider a case such that the Hamiltonian analysis gives k constraints (i.e. the dimension of M_F is $2n-k$), of which m_1 of them form the set of primary constraints, and a certain number of these, let us say r, are the final primary first class constraints.

The number of independent Lagrangian constraints of the form (22), i.e. the maximal set of projectable constraints, is $k-m_1$ because $FL^*\phi$ is identically zero if ϕ is a primary constraint whereas for secondary constraints $FL^*\phi$ gives independent functions.

The rest of the Lagrangian constraints (23), i.e. the strictly non-projectable ones, number m_1-r. Thus we have the following theorem.

Theorem 2. The number of independent Lagrangian constraints is $k-r$, k being the number of independent Hamiltonian constraints and r the number of final primary first class constraints.

We conclude therefore that, in the general case, the final surfaces of motion $M_F \subset T^*Q$, $S_F \subset TQ$ do not have the same dimensions. This fact does not contradict the equivalence between the equations of motion in both formalisms, which was proved in [2]. We may wonder whether the number of degrees of freedom is different in one or another formalism. The answer is *no*: it is possible to prove [5] that, when the gauge invariances are taken into account and superfluous variables are eliminated, the number of true degrees of freedom is the same.

4. The arbitrary functions

The evolution operator in $S_1 \subset TQ$ is (2)

$$D_L = \mathring{D} + \eta^{\mu_f}(t)\Gamma_{\mu_f} + \eta^{\bar{\mu}_f}(t)\Gamma_{\bar{\mu}_f}.$$

The action of D_L over Lagrangian constraints must lead to new constraints or to the determination in TQ of some arbitrary functions $\eta(t)$. The classification of constraints given by (22) and (23) is very suited to this end.

Let us consider (23). Tangency of D_L to the constraint $\chi^{\bar{\mu}_f}$ in S_F means

$$0 \approx D_L\chi^{\bar{\mu}_f} = (D_L \circ FL^*)\lambda^{\bar{\mu}_f} - D_L v^{\bar{\mu}_f}$$

where $\approx$ means equality on S_F.

But we know from (16) $D_L \circ FL^* =_{S_1} K$. We also know $\Gamma_\mu \circ FL^* = 0$ and $\Gamma_\mu v^\nu = \delta^\nu_\mu$. Then, taking into account the appendix, we get

$$D_L v^{\bar{\mu}_f} \underset{S_1}{=} \dot{q}\frac{\partial v^{\bar{\mu}_f}}{\partial q} - \eta^{\bar{\mu}_f}.$$

Therefore we arrive at the determination of $\eta^{\bar{\mu}_f}$ (on S_F):

$$\eta^{\bar{\mu}_f}(t) \to f^{\bar{\mu}_f}(q, \dot{q}) = K\lambda^{\bar{\mu}_f} - \dot{q}\frac{\partial v^{\bar{\mu}_f}}{\partial q}. \tag{24}$$

So, stability of the non-projectable constraints (23) leads simply to the determination in TQ of the arbitrary functions $\eta^{\bar{\mu}_f}(t)$.

Let us now consider the action of D_L on the projectable constraints. Recalling (16) we have

$$D_L(FL^*\phi) \underset{S_1}{=} K\phi. \tag{25}$$

Therefore, as a first consequence, $K\phi = 0$ are Lagrangian constraints. In fact, they are *all* the Lagrangian constraints. This can be seen by considering that the stability of the non-projectable constraints $\chi^{\bar{\mu}_f}$ does not give new constraints, whereas the stability of the projectable ones is, as we see from (25), just $K\phi = 0$. Thus, all the secondary Lagrangian constraints can be written in the form $K\phi = 0$. On the other hand, we know [2] that the primary Lagrangian constraints, $(\alpha, \gamma_\mu) = 0$, satisfy

$$(\alpha, \gamma_\mu) = K\phi^{(0)}_\mu.$$

Therefore we have the following.

Theorem 3. All the Lagrangian constraints can be written in the form

$$K\phi = 0 \qquad \phi \in \mathscr{C}.$$

5. From Hamilton–Dirac to Euler–Lagrange equations

The evolution operator $K: \Lambda^0(T^*Q) \to \Lambda^0(TQ)$ has produced a deep insight into the connection of the Hamiltonian and Lagrangian formalisms for constrained systems. Let us show how powerful this operator is in obtaining the final Euler-Lagrange equations (on S_F) from the final Hamilton-Dirac equations (on M_F).

The final evolution operator in T^*Q can be written as

$$D_{\rm H} \underset{M_{\rm F}}{=} \{-, \mathscr{H}_{\rm c}\} + \lambda^{\bar{\mu}_{\rm f}}(q,p)\{-, \phi^{(0)}_{\bar{\mu}_{\rm f}}\} + u^{\mu_{\rm f}}(t)\{-, \phi^{(0)}_{\mu_{\rm f}}\}$$

with $u^{\mu_{\rm f}}(t)$ arbitrary functions of t.

An integral curve $(q(t), p(t)) \in M_{\rm F}$ of $D_{\rm H}$ will satisfy

$$\mathrm{d}q/\mathrm{d}t = (D_{\rm H} q) =: g(q, p, t)$$

and the accelerations will be obtained by

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{\partial g}{\partial q}\dot{q} + \frac{\partial g}{\partial p}\dot{p} + \frac{\partial g}{\partial t}$$

but since we are on a curve $(q(t), p(t))$ which is a solution of the equations of motion, we can replace p by $\mathfrak{P}(q, \dot{q})$ (we apply FL^*) and also $\dot{p}$ by $\partial L/\partial q$. Therefore we arrive at

$$\begin{aligned}\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} &= \dot{q} FL^* \frac{\partial g}{\partial q} + \frac{\partial L}{\partial q} FL^* \frac{\partial g}{\partial p} + FL^* \frac{\partial g}{\partial t} \\ &= Kg + FL^* \frac{\partial g}{\partial t}. \qquad (26)\end{aligned}$$

We are considering here variables q, p satisfying the constraints $\phi = 0$, $\forall \phi \in \mathscr{C}$, so the variables $(q, \dot{q})$ in (26) satisfy the Lagrangian constraints $K\phi = 0$, $\forall \phi \in \mathscr{C}$. Bearing this in mind, let us calculate (26):

$$\begin{aligned}\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \underset{S_{\rm F}}{=} & K\{q, \mathscr{H}_{\rm c}\} + (K\lambda^{\bar{\mu}_{\rm f}}) FL^*\{q, \phi^{(0)}_{\bar{\mu}_{\rm f}}\} + (FL^*\lambda^{\bar{\mu}_{\rm f}}) K\{q, \phi^{(0)}_{\bar{\mu}_{\rm f}}\} \\ & + u^{\mu_{\rm f}}(t) K\{q, \phi^{(0)}_{\mu_{\rm f}}\} + \frac{\mathrm{d}u^{\mu_{\rm f}}}{\mathrm{d}t}\{q, \phi^{(0)}_{\mu_{\rm f}}\} \qquad (27)\end{aligned}$$

but $FL^*\lambda^{\bar{\mu}_{\rm f}} = v^{\bar{\mu}_{\rm f}}$ on $S_{\rm F}$ and $u^{\mu_{\rm f}}(t) = v^{\mu_{\rm f}}(q, \dot{q})$ on a motion $q(t)$. Therefore

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \underset{S_{\rm F}}{=} K\{q, \mathscr{H}_{\rm c}\} + v^{\mu}(q, \dot{q}) K\{q, \phi^{(0)}_{\mu}\} + (K\lambda^{\bar{\mu}_{\rm f}})\gamma_{\bar{\mu}_{\rm f}} + \frac{\mathrm{d}u^{\mu_{\rm f}}}{\mathrm{d}t}\gamma_{\mu_{\rm f}} \qquad (28)$$

and using the explicit form of K:

$$\begin{aligned}\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \underset{S_{\rm F}}{=} & \dot{q}\left(FL^* \frac{\partial^2 \mathscr{H}_{\rm c}}{\partial q\, \partial p} + v^{\mu}(q, \dot{q}) FL^* \frac{\partial \phi^{(0)}_{\mu}}{\partial q\, \partial p}\right) \\ & + \frac{\partial L}{\partial q}\left(FL^* \frac{\partial^2 \mathscr{H}_{\rm c}}{\partial p\, \partial p} + v^{\mu}(q, \dot{q}) FL^* \frac{\partial \phi^{(0)}_{\mu}}{\partial p\, \partial p}\right) + (K\lambda^{\bar{\mu}_{\rm f}})\gamma_{\bar{\mu}_{\rm f}} + \frac{\mathrm{d}u^{\mu_{\rm f}}}{\mathrm{d}t}\gamma_{\mu_{\rm f}}. \qquad (29)\end{aligned}$$

Note the presence of the piece M of (4) in (29). On the other hand, derivation with respect to q of the Lagrangian identity (8) gives

$$FL^* \frac{\partial^2 \mathscr{H}_{\rm c}}{\partial q\, \partial p} + v^{\mu}(q, \dot{q}) FL^* \frac{\partial^2 \phi^{(0)}_{\mu}}{\partial q\, \partial p} = -\frac{\partial \mathfrak{P}}{\partial q} M - \frac{\partial v^{\mu}}{\partial q}\gamma_{\mu}$$

and substitution in (29)

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} \underset{S_{\rm F}}{=} \dot{q}\left(-\frac{\partial \mathfrak{P}}{\partial q} M\right) + \frac{\partial L}{\partial q} M + K\lambda^{\bar{\mu}_{\rm f}}\gamma_{\bar{\mu}_{\rm f}} + \frac{\mathrm{d}u^{\mu_{\rm f}}}{\mathrm{d}t}\gamma_{\mu_{\rm f}} - \dot{q}\frac{\partial v^{\mu}}{\partial q}\gamma_{\mu}$$

or

$$\left.\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}\right|_{S_F} = (\alpha M) + \left(K\lambda^{\bar{\mu}_f} - \dot{q}\frac{\partial v^{\bar{\mu}_f}}{\partial q}\right)\gamma_{\bar{\mu}_f} + \left(\frac{\mathrm{d}u^{\mu_f}}{\mathrm{d}t} - \dot{q}\frac{\partial v^{\mu_f}}{\partial q}\right)\gamma_{\mu_f}. \tag{30}$$

But in the previous section we determined the functions

$$f^{\bar{\mu}_f} = K\lambda^{\bar{\mu}_f} - \dot{q}\frac{\partial v^{\bar{\mu}_f}}{\partial q}.$$

Thus equations (30) are just the final Euler–Lagrange equations of motion (see [2], § 4)

$$\left.\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}\right|_{S_F} = (\alpha M) + f^{\bar{\mu}_f}\gamma_{\bar{\mu}_f} + \eta^{\mu_f}(t)\gamma_{\mu_f}. \tag{31}$$

As a by-product of (30) we have obtained the relation between the remaining arbitrary functions u^{μ_f} of the Hamiltonian and Lagrangian formalisms:

$$\eta^{\mu_f} = \frac{\mathrm{d}u^{\mu_f}}{\mathrm{d}t} - \dot{q}\frac{\partial v^{\mu_f}}{\partial q}. \tag{32}$$

6. Conclusions

In this work we have pursued the study of the connection between the Lagrangian and Hamiltonian formalisms for constrained systems begun in [1, 2]. The relevance of the evolution operator $K: \Lambda^0(T^*Q) \to \Lambda^0(TQ)$ introduced in [2] is emphasised through several applications. Thus we show that all the Lagrangian constraints can be obtained by applying K to the Hamiltonian constraints. The operator K has also been used to connect the so-called arbitrary functions of Hamiltonian and Lagrangian formalisms and also to give a procedure to derive the final form of the Euler–Lagrange equations of motion (in a normal form on the surface of constraints) from the final Hamilton–Dirac equations of motion. We have also presented a new form for the Lagrangian constraints, part of which are the pull-back of the Hamiltonian constraints and the rest come from the elimination, due to consistency requirements, of part of the initial arbitrariness of the Hamilton–Dirac time evolution operator. This classification of Lagrangian constraints separates the maximal set of projectable Lagrangian constraints from the set of strictly non-projectable constraints. Owing to this separation, we can calculate the dimension of the final surface of Lagrangian constraints. It turns to be

$$\dim S_F = \dim M_F + r$$

r being the number of final primary first class Hamiltonian constraints.

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Appendix

We derive some results used in §§ 2 and 4. From the identity (8):

$$\dot{q} = FL^*\{q, \mathscr{H}_c\} + v^\mu(q, \dot{q}) FL^*\{q, \phi_\mu^{(0)}\}$$

the derivation with respect to $\dot{q}$ gives

$$\begin{aligned}\delta &= FL^* \frac{\partial^2 \mathscr{H}_c}{\partial p\,\partial p} \frac{\partial \mathfrak{P}}{\partial \dot{q}} + v^\mu FL^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial p\,\partial p} \frac{\partial \mathfrak{P}}{\partial \dot{q}} + \frac{\partial v^\mu}{\partial \dot{q}} \gamma_\mu \\ &= \left(FL^* \frac{\partial^2 \mathscr{H}_c}{\partial p\,\partial p} + v^\mu FL^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial p\,\partial p} \right) \frac{\partial \mathfrak{P}}{\partial \dot{q}} + \frac{\partial v^\mu}{\partial \dot{q}} \gamma_\mu .\end{aligned}$$

If we remember that

$$\partial \mathfrak{P} / \partial \dot{q} = W$$

and we define (4)

$$M = FL^* \frac{\partial^2 \mathscr{H}_c}{\partial p\,\partial p} + v^\mu FL^* \frac{\partial^2 \phi_\mu^{(0)}}{\partial p\,\partial p}$$

we have

$$\delta_i^j = M^{jk} W_{ki} + \frac{\partial v^\mu}{\partial \dot{q}^i} \gamma_\mu^j . \tag{A1}$$

This is an important completeness relation which has been used to construct the time evolution Lagrangian operator (2).

A consequence of this completeness relation will now be deduced.

Contraction of (A1) with $\partial v^\nu / \partial \dot{q}^j$ gives

$$\frac{\partial v^\nu}{\partial \dot{q}^i} = W_{ik} M^{kj} \frac{\partial v^\nu}{\partial \dot{q}^j} + \frac{\partial v^\mu}{\partial \dot{q}^i} \gamma_\mu^j \frac{\partial v^\nu}{\partial \dot{q}^j} .$$

But

$$\gamma_\mu^j \frac{\partial v^\nu}{\partial \dot{q}^j} \equiv \Gamma_\mu v^\nu = \delta_\mu^\nu .$$

Thus

$$W_{ik} M^{kj} \frac{\partial v^\nu}{\partial \dot{q}^j} = 0 .$$

We know a basis, γ_μ, $\mu = 1, \ldots, m_1$, of the null vectors of W. Therefore

$$M^{kj} \frac{\partial v^\nu}{\partial \dot{q}^j} = \eta^{\nu\mu} \gamma_\mu^k$$

and then

$$(\alpha M)^j \frac{\partial v^\nu}{\partial \dot{q}^j} = \alpha_k M^{kj} \frac{\partial v^\nu}{\partial \dot{q}^j} = \eta^{\nu\mu} (\alpha \gamma_\mu)$$

i.e.

$$(\alpha M)\frac{\partial}{\partial \dot{q}} v^{\nu} \underset{S_1}{=} 0. \tag{A2}$$

This is the result used in § 4.

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